Lecture 7: Vector Spaces III - Eigenvalues and Eigenvectors

1. Key points

- Eigenvalues and eigenvectors
- If there is no degeneracy in eigenvalues, the corresponding eigenvectors are orthogonal.
- If the operator is Hermitian, its eigenvalues are all real.
- Eigenvectors form a complete basis.

Maple commands

- LinearAlgebra package
- Determinant
- solve
- Eigenvalues
- Eigenvectors
- Norm
- Normalize

2. Eigenvalues and Eigenvectors

In general, \( A \) transforms a vector to another vector as \( |D_{A}| \). Some vectors are transformed to vectors parallel to themselves. That means \( A|\alpha| = \lambda|\alpha| \) are called eigenvector and eigenvalue of \( A \), respectively.

- If an eigenvector is known, we can find the corresponding eigenvalue immediately,
  \( \lambda = \frac{\langle \alpha | A | \alpha \rangle}{\langle \alpha | \alpha \rangle} \).
- In a matrix representation, the eigenvalue problem is \( A|\alpha| = \lambda|\alpha| \) where \( A \) is a matrix and \( |\alpha| \) is a vector.
- Nontrivial solutions exist if and only if \( |A-\lambda| = 0 \). This characteristic equation determines the eigenvalues.
- When \( A \) is a rank \( N \) matrix, there are in general \( N \) eigenvalues and \( N \) corresponding eigenvectors.
- Once the eigenvalues are found, a solution to the simultaneous equation \( (A-\lambda)|\alpha| = 0 \) is the corresponding eigenvector.
- Note that the eigenvector cannot be determined uniquely. If \( |\alpha| \) is an eigenvector, \( c|\alpha| \) is also an eigenvector.
- For convenience, the eigenvectors are usually normalized as \( \langle \alpha | \alpha \rangle = 1 \).

Example: Matrix representation

Find all eigenvalues of matrix
\[
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}
\]
and corresponding normalized eigenvectors.
There are two eigenvalues. \( \lambda[1] = 1 \) and \( \lambda[2] = -1 \).

Let us express the eigenvector as

\[
\mathbf{v} := \begin{bmatrix} x \\ y \end{bmatrix},
\]

We construct the characteristic equation for the first eigenvalue is

\[
E_Q := (\mathcal{A} - \lambda\mathcal{A}) \cdot \mathbf{v} = 0
\]

The solution is simply \( \mathbf{v} = \begin{bmatrix} -x + y \\ x - y \end{bmatrix} \). Hence, the eigenvector is

\[
\mathbf{v} := \begin{bmatrix} x \\ y \end{bmatrix}.
\]

The remaining unknown is determined by normalization.

\[
solve(Norm(\mathbf{v}, 2) = 1, x)
\]

Either solution gives the normalized eigenvector.

\[
\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ or } \frac{-1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

These vectors are not linearly independent to each other. You can pick a convenient one. We pick the first one

\[
v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

Similarly for the second eigenvalue \( \lambda[2] = -1 \).

\[
E_Q := (\mathcal{A} - \lambda\mathcal{A}) \cdot \mathbf{v} = 0
\]
The solution is simply \( x = -y \).

Hence, the eigenvector is

\[
\mathbf{v} := \begin{bmatrix} x \\ -x \end{bmatrix}.
\]

The remaining unknown is determined by normalization.

\[
solve(Norm(\mathbf{v}, 2) = 1, x)
\]

\[
\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}
\]

Using the first solution, \( \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).

**Example: Function representation**

Find eigenvalues and eigenvectors for operator \( \mathcal{A} = -\frac{d^2}{dx^2} \) assuming that the eigenfunction \( u(x) \) is defined on \( x \in [-a, a] \) and the boundary value is given by \( u(\pm a) = 0 \).

**Answer**

The eigenvalue is given by

\[
\mathcal{A}|u = \lambda |u \quad \Rightarrow \quad -\frac{d^2}{dx^2} u(x) = \lambda u(x).
\]

This is equivalent to the Newton equation for a simple harmonic oscillator:

\[
\frac{d^2}{dt^2} x(t) = -\omega^2 x(t) \quad \text{with} \quad \lambda = \omega^2.
\]

The solution to this equation is well known. \( x(t) \)

\[
= A \cos(\omega t) + B \sin(\omega t).
\]

Hence, the solution to the eigenvalue is \( u(x) \)

\[
= A \cos(\sqrt{\lambda} x) + B \sin(\sqrt{\lambda} x).
\]

The value of \( A, B, \lambda \) are determined by the boundary conditions:

\[
A \cos(\sqrt{\lambda} \cdot a) + B \sin(\sqrt{\lambda} \cdot a) = 0
\]

\[
A \cos(\sqrt{\lambda} \cdot a) + B \sin(\sqrt{\lambda} \cdot a) = 0
\]

\[
A \cos(-\sqrt{\lambda} \cdot a) + B \sin(-\sqrt{\lambda} \cdot a) = 0
\]

\[
A \cos(\sqrt{\lambda} \cdot a) - B \sin(\sqrt{\lambda} \cdot a) = 0
\]

Adding (8) and (9), we have \( A \cos(\sqrt{\lambda} \cdot a) = 0 \). This leads to \( A = 0 \) or

\[
\sqrt{\lambda} \cdot a = \left( n + \frac{1}{2} \right) \pi \quad \text{where} \quad n \text{ is integer}.
\]

Now examine these two possibilities individually.

If \( A = 0 \), the eigenfunction must be \( u(x) = B \sin(\sqrt{\lambda} x) \). Applying the boundary condition
\[ u(\pm a) = 0, \]
\[ \sin(\sqrt{\lambda} \ a) = 0. \text{ (} x = -a \text{ leads to the same equation.)} \]
Solving the equation, we find
\[ \sqrt{\lambda} \ a = n \pi \text{ where } n \in \mathbb{Z} \] and thus
\[ \lambda_n = n^2 \left( \frac{\pi}{a} \right)^2 \]
and the corresponding eigenfunction is
\[ u(x) = B \sin \left( n \frac{\pi x}{a} \right). \]

If \( A \neq 0 \), then
\[ \sqrt{\lambda} a = \left( n + \frac{1}{2} \right) \pi. \]
Substituting this into the boundary conditions (8) and (9), we find \( B = 0 \).

Now, we found the eigenvalues
\[ \lambda_n = \left( n + \frac{1}{2} \right)^2 \left( \frac{\pi}{a} \right)^2 \]
and the corresponding eigenfunction is
\[ u(x) = A \cos \left( \left( n + \frac{1}{2} \right) \frac{\pi x}{a} \right). \]

The constants \( A \) and \( B \) are determined by normalization. For \( A \),
\[ \langle u | u \rangle = \int_{-a}^{a} |u(x)|^2 \, dx = |A|^2 \int_{-a}^{a} \sin^2 \left( n \frac{\pi x}{a} \right) \, dx = 1 \rightarrow |A|^2 a = 1 \]
Hence, \( A = \frac{1}{\sqrt{a}} \).

Similarly for \( B \),
\[ |B|^2 \int_{-a}^{a} \cos^2 \left( \left( n + \frac{1}{2} \right) \frac{\pi x}{a} \right) \, dx = 1 \rightarrow |B|^2 a = 1 \]
Hence, \( B = \frac{1}{\sqrt{a}} \).

In summary, eigenvalues and the corresponding eigenfunctions of
\[ -\frac{d^2}{dx^2} u(x) = \lambda \ u(x) \]
are:

For eigenvalue
\[ \lambda_n = n^2 \left( \frac{\pi}{a} \right)^2 \]
the eigenfunction is
\[ u_n(x) = \frac{1}{\sqrt{a}} \sin \left( n \frac{\pi x}{a} \right), \]
and

For eigenvalue
\[ \lambda_n = \left( n + \frac{1}{2} \right)^2 \left( \frac{\pi}{a} \right)^2 \]
the eigenfunction is
\[ u_n(x) = \frac{1}{\sqrt{a}} \cos \left( \left( n + \frac{1}{2} \right) \frac{\pi x}{a} \right). \]

**Common sense: Diagonal Matrix**

If the matrix representation of the operator is a diagonal matrix, its eigenvalues are diagonal elements.
Diagonal matrix
\[
\begin{pmatrix}
  a & 0 & 0 \\
  0 & b & 0 \\
  0 & 0 & c
\end{pmatrix}
\]
has eigenvalues \(\lambda_1 = a, \ \lambda_2 = b,\) and \(\lambda_3 = c.\) The corresponding eigenvectors are
\[
\begin{pmatrix}
  1 \\
  0 \\
  0
\end{pmatrix}, \quad
\begin{pmatrix}
  0 \\
  1 \\
  0
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
  0 \\
  0 \\
  1
\end{pmatrix}.
\]

Example:
\[
\begin{pmatrix}
  1 & 0 \\
  0 & -1
\end{pmatrix}
\]
has eigenvalues 1 and -1.

The corresponding eigenvectors are
\[
\begin{pmatrix}
  1 \\
  0
\end{pmatrix}\quad \text{and}\quad \begin{pmatrix}
  0 \\
  1
\end{pmatrix}.
\]

Maple example:

Find eigenvalues and the corresponding eigenvectors of a matrix
\[
\begin{pmatrix}
  2 & -1 & 0 \\
  -1 & 2 & -1 \\
  0 & -1 & 2
\end{pmatrix}
\]

To find all eigenvalues:

\[\text{restart} :\]
\[\text{with(LinearAlgebra)} :\]
\[A := \begin{pmatrix}
  2 & -1 & 0 \\
  -1 & 2 & -1 \\
  0 & -1 & 2
\end{pmatrix} :\]
\[\lambda := \text{Eigenvalues}(A)\]
\[
\begin{pmatrix}
  2 \\
  2 - \sqrt{2} \\
  2 + \sqrt{2}
\end{pmatrix}
\]
(10)

The Maple output indicates three eigenvalues: \(\lambda_1 = 2, \ \lambda_2 = 2 - \sqrt{2},\) and \(\lambda_3 = 2 + \sqrt{2}.

To find eigenvalues and corresponding eigenvectors:
\[v := \text{Eigenvectors}(A)\]
\[
\begin{pmatrix}
  2 + \sqrt{2} \\
  2 - \sqrt{2} \\
  2
\end{pmatrix}, \quad
\begin{pmatrix}
  1 & 1 & -1 \\
  -\sqrt{2} & \sqrt{2} & 0 \\
  1 & 1 & 1
\end{pmatrix}
\]
(11)
The first matrix is the eigenvalues which is the same as the output of `Eigenvalues(A)`. The corresponding eigenvectors are columns of the second matrix.

The eigenvector for $\lambda_1$ is

\[ u_1 := v[2][1 .. 3, 1] \]

\[ u_1 := \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} \quad (12) \]

The eigenvector for $\lambda_2$ is

\[ u_2 := v[2][1 .. 3, 2] \]

\[ u_2 := \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} \quad (13) \]

The eigenvector for $\lambda_3$ is

\[ u_3 := v[2][1 .. 3, 3] \]

\[ u_3 := \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad (14) \]

3. Eigenvectors of an operator form an orthonormal basis set

Suppose an operator $A$ has $n$ eigenvectors: $A | i \rangle = \lambda_i | i \rangle$ ($i = 1 \cdots n$).

If $\lambda_i \neq \lambda_j$, then $\langle i | j \rangle = 0$.

**Maple example**

First, we check the orthogonality.

\[ \text{DotProduct}(u_1, u_2) = 0 \]
\[ \text{DotProduct}(u_2, u_3) = 0 \]
\[ \text{DotProduct}(u_3, u_1) = 0 \]

Yes, they are orthogonal.

Are they normalized?

\[ \text{Norm}(u_1, 2) = 2 \]
\[ \text{Norm}(u_2, 2) = 2 \]
\[ \text{Norm}(u_3, 2) = \sqrt{2} \]

No. Maple does not normalize it. You must normalize by yourself.
Common sense
If an operator is expressed in a matrix form using its eigenvectors as base vectors, it is a diagonal matrix and its matrix elements are its eigenvalues.

If \( A | \alpha_j \rangle = \lambda_j | \alpha_j \rangle \) and \( \langle \alpha_j | \alpha_j \rangle = \delta_j \), then \( A | \alpha_j \rangle = \langle \alpha_j | A | \alpha_j \rangle = \lambda_j \langle \alpha_j | \alpha_j \rangle = \lambda_j \delta_j \).

For \( \mathcal{A} \) is in a 3-dimensional Hilbert space, \( A^\dagger = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \).

4. Eigenvalues of Hermitian operators are all real

Common sense
If \( \mathcal{A} \) is Hermitian (self-adjoint) operator, its eigenvalues are all real.

Maple example
restart:
with(LinearAlgebra):

\[
\begin{align*}
\text{Normalize} (u_1, \text{Euclidean}) = & \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \sqrt{2} \\ \frac{1}{2} \end{bmatrix} \\
\text{Normalize} (u_2, \text{Euclidean}) = & \begin{bmatrix} \frac{1}{2} \sqrt{2} \\ \frac{1}{2} \\ -\frac{1}{2} \sqrt{2} \end{bmatrix} \\
\text{Normalize} (u_3, \text{Euclidean}) = & \begin{bmatrix} \frac{1}{2} \sqrt{2} \\ 0 \\ \frac{1}{2} \sqrt{2} \end{bmatrix}
\end{align*}
\]
Consider an operator in a matrix representation: \( \mathcal{A} := \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix} \)

Is \( \mathcal{A} \) self-adjoint?

\[ \mathcal{A}^\dagger \mathcal{A} \]

Hence \( \mathcal{A} = \mathcal{A}^\dagger \). (Self-adjoint) This suggests that eigenvalues are all real.

\( \lambda := \text{Eigenvalues}(\mathcal{A}) \)

\[
\lambda := \left[ \left( \frac{8 + 3 \sqrt[3]{237}}{3} \right)^{1/3} + \frac{13}{3 \left( 8 + 3 \sqrt[3]{237} \right)^{1/3}} + \frac{5}{3} \right] \\
- \left( \frac{8 + 3 \sqrt[3]{237}}{6} \right)^{1/3} - \frac{13}{6 \left( 8 + 3 \sqrt[3]{237} \right)^{1/3}} + \frac{5}{3} \\
\sqrt{3} \left( \frac{8 + 3 \sqrt[3]{237}}{3} \right)^{1/3} \frac{13}{3 \left( 8 + 3 \sqrt[3]{237} \right)^{1/3}} \\
+ \frac{2}{2} \right] \\
- \left( \frac{8 + 3 \sqrt[3]{237}}{6} \right)^{1/3} - \frac{13}{6 \left( 8 + 3 \sqrt[3]{237} \right)^{1/3}} + \frac{5}{3} \\
\sqrt{3} \left( \frac{8 + 3 \sqrt[3]{237}}{3} \right)^{1/3} \frac{13}{3 \left( 8 + 3 \sqrt[3]{237} \right)^{1/3}} \\
- \frac{2}{2} \right] \\
\] (16)

The above expression includes complex numbers. That does not mean that the values are complex. It looks complicated but can be simplified.

Let Maple do it.

\( \text{evalc}(\lambda[1]) \)

\[
\frac{2 \sqrt{13}}{3} \cos \left( \arctan \left( \frac{3 \sqrt{237}}{8} \right) \right) + \frac{5}{3}
\]
Indeed, they are all real.

Now, consider

\[ \lambda_2 = \frac{\sqrt{13} \cos \left( \arctan \left( \frac{3 \sqrt{237}}{8} \right) \right)}{3} + \frac{5}{3} \]  
\[ \lambda_3 = \frac{\sqrt{13} \sin \left( \arctan \left( \frac{3 \sqrt{237}}{8} \right) \right)}{3} + \frac{5}{3} \]  

(18)

(19)

Indeed, they are all real.

Now, consider

\[ B := \begin{bmatrix} 0 & 3 - I \\ 3 - I & 0 \end{bmatrix} \]

(20)

\[ B^{\%H} = \begin{bmatrix} 0 & 3 + I \\ 3 + I & 0 \end{bmatrix} \]

Hence \( B \neq B^\dagger \). (Not self-adjoint) This suggests that eigenvalues can be complex.

**Eigenvalues** \( B \)

\[ \begin{bmatrix} 3 - I \\ -3 + I \end{bmatrix} \]

(21)

Yes, they are indeed complex.

## 5. Two Operators that commute share the same eigenvectors

**Common sense**

If \( \mathcal{A}|u\rangle = \lambda |u\rangle \) and \([\mathcal{A}, \mathcal{B}] = 0\), then \( \mathcal{B}|u\rangle = \mu |u\rangle \).

where \([\mathcal{A}, \mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}\).

This means that when \( \mathcal{A} \) and \( \mathcal{B} \) commute, they share the same eigenvectors. However, their eigenvalues do not have to be the same.

**Maple example**

```
restart:
with(LinearAlgebra):

Consider two operators in matrix representation, \( A := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \) and \( B := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).
```
Check if they commute.

\[ A.B - B.A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]

Since they compute, they must have the same eigenvectors.

**Eigenvectors** \((A)\)

\[
\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\]  \hspace{1cm} (22)

**Eigenvectors** \((B)\)

\[
\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\]  \hspace{1cm} (23)

Both operators have eigenvector \(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\) and \(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\). However, they don't share the eigenvalues.

**Homework: Due 9/25 11am**

**7.1**

Find all eigenvalues and corresponding normalized eigenvectors of the following matrix

\[
A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}
\]

**7.2**

From the following matrices, find matrices whose eigenvalues are all real.

- (a) \[
\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 3 \end{bmatrix}
\]
- (b) \[
\begin{bmatrix} i & 3 & 2 \\ 3 & -4 & 1 \\ 2 & 1 & 2i \end{bmatrix}
\]
- (c) \[
\begin{bmatrix} -2 & i & 1 \\ -i & 0 & 2 - i \\ 1 & 2 + i & 3 \end{bmatrix}
\]
- (d) \[
\begin{bmatrix} 3 & 2 + i & -i \\ 2 + i & 2 & -2 \\ -i & -2 & 1 \end{bmatrix}
\]

**7.3**

A linear operator \(\mathcal{L} = \frac{d}{dx} + 2 x\) has eigenvalue \(\lambda = 0\). That means \(\mathcal{L} f(x) = 0\) where \(f(x)\) is the corresponding eigenfunction defined in \(x \in \mathbb{R}\). The eigenfunction should satisfy boundary condition \(\lim_{x \to \pm \infty} f(x) = 0\). Find \(f(x)\) and normalized it.