PH420/520 Test 3 Solution

```
ClearAll["Gloabal`*"]; $Line = 0;
```

In-Class Test

1(a) Separating the variables,

$$\frac{dy}{y^2} = x\, dx \tag{1}$$

Integrating it, we obtain

$$\int \frac{1}{y^2}\, dy = \int x\, dx \longrightarrow \frac{-1}{y} = \frac{1}{2} x^2 + C \tag{2}$$

Inverting it, we find the solution

$$y = -\frac{2}{x^2 + 2\, C} \tag{3}$$

where C is a constant (You can replace 2 C with a new constant.)

Check it with Mathematica

```
DSolve[y'[x] == x y[x]^2, y[x], x]
```

$$\left\{\left\{y[x] \to -\frac{2}{x^2 + 2\, C[1]}\right\}\right\}$$

1(b) Separating the variables,

$$\frac{dy}{y} = \frac{dx}{x+1} \tag{4}$$

Integrating it, we obtain

$$\int \frac{1}{y}\, dy = \int (x+1)\, dx \longrightarrow \ln y = \frac{1}{2} x^2 + x + A \tag{5}$$

Inverting it, we find the solution

$$y = C \exp\left(\frac{x^2}{2} + x\right) \tag{6}$$

where $C = e^A$ is a constant.

Check it with Mathematica

```
DSolve[y'[x] == x y[x] + y[x], y[x], x]
```

$$\left\{\left\{y[x] \to e^{x+\frac{x^2}{2}}\, C[1]\right\}\right\}$$

1(C) The characteristic equation is

$$\lambda^2 + 2\lambda + 5 = 0 \tag{7}$$

and its solutions are

$$\lambda = -1 \pm \sqrt{1-5} = -1 \pm 2\, i \tag{8}$$

The general solution to the ODE is

$$y = e^{-x}\left(A_1 e^{2ix} + A_2 e^{-2ix}\right) = \mathbf{e^{-x}\left[C_1 \cos(2x) + C_2 \sin(2x)\right]} \quad (9)$$

where C_1 and C_2 are constants.

Check it with Mathematica

```
DSolve[y''[x] + 2 y'[x] + 5 y[x] == 0, y[x], x]
{{y[x] → e^-x C[2] Cos[2 x] + e^-x C[1] Sin[2 x]}}
```

Check it with Mathematica

Check it Mathematica with

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2. Let $x(t) = T(t) - T_R$. Then, the ODE is written as

$$\frac{dx}{dt} = -k\,x \quad (10)$$

and its solution is

$$x(t) = x(0)\, e^{-kt} \quad (11)$$

Moving back to the original variable,

$$T(t) - T_R = [T(0) - T_R]\, e^{-kt} \quad (12)$$

Substituting the initial temperature $T(0) = T_0$, we obtain

$$\mathbf{T(t) = (T_0 - T_R)\, e^{-kt} + T_R} \quad (13)$$

As $t \longrightarrow \infty$, the first term in the right hand side vanishes. Thus

$$\mathbf{\lim_{t\to\infty} T(t) = T_R} \quad (14)$$

Check it with Mathematica

```
sol = DSolve[{T'[t] == -k (T[t] - TR), T[0] == T0}, T[t], t]
{{T[t] → e^-k t (T0 - TR + e^k t TR)}}

Assuming[k > 0, Limit[T[t] /. sol, t → ∞]]
{TR}
```

---

3. Divide the ODE by m and let y(t)=x(t)+mg/k, the ODE becomes

$$\frac{d^2y}{dt^2} = -\frac{k}{m}\, y \quad (15)$$

This is a simple harmonic oscillator and its solution is

$$y(t) = A\cos(\omega t) + B\sin(\omega t) \quad (16)$$

where $\omega=\sqrt{k/g}$. The initial values are

$$y(0) = x(0) + mg/k = A \quad \text{and} \quad y'(0) = x'(0) = \omega B \quad (17)$$

Since x(0)=0 and $\dot{x}(0)$=0, A=mg/k and B=0. Then, the solution is

$$x(t) = \frac{mg}{k}\cos\omega t - \frac{mg}{k} = \mathbf{\frac{g}{\omega^2}(\cos\omega t - 1)} \quad (18)$$

Check it with Mathematica

```
Assuming[m > 0 && k > 0, DSolve[{m x''[t] == -m g - k x[t], x[0] == 0, x'[0] == 0}, x[t], t]]
```

$$\left\{\left\{x[t] \to \frac{-g\,m + g\,m\,\mathrm{Cos}\left[\sqrt{\frac{k}{m}}\;t\right]}{k}\right\}\right\}$$

$$\left\{\left\{x[t] \to \frac{-g\,m + g\,m\,\mathrm{Cos}\left[\sqrt{\frac{k}{m}}\;t\right]}{k}\right\}\right\} \;/.\; k \to m\,\omega^2$$

$$\left\{\left\{x[t] \to \frac{-g\,m + g\,m\,\mathrm{Cos}\left[t\,\sqrt{\omega^2}\right]}{m\,\omega^2}\right\}\right\}$$

$$\left\{\left\{x[t] \to \frac{-g\,m + g\,m\,\mathrm{Cos}\left[t\,\sqrt{\omega^2}\right]}{m\,\omega^2}\right\}\right\} \;//\; \mathrm{Simplify}$$

$$\left\{\left\{x[t] \to \frac{g\left(-1 + \mathrm{Cos}\left[t\,\sqrt{\omega^2}\right]\right)}{\omega^2}\right\}\right\}$$

---

4. The characteristic equation is

$$L\,\lambda^2 + R\,\lambda + \frac{1}{C} = 0 \tag{19}$$

and its solution is

$$\lambda = \frac{-R \pm \sqrt{R^2 - 4\,\frac{L}{C}}}{2\,L} = -\frac{R}{2\,L} \pm \sqrt{\left(\frac{R}{2\,L}\right)^2 - \frac{1}{CL}} \tag{20}$$

Since J decays with oscillation,

$$\left(\frac{R}{2\,L}\right)^2 - \frac{1}{CL} < 0 \tag{21}$$

Let $\Omega^2 = \frac{1}{CL} - \left(\frac{R}{2L}\right)^2$

A general solution to the ODE is

$$J\,(t) = e^{-R\,t/2\,L}\left(C_1\,e^{i\,\Omega\,t} + C_2\,e^{-i\,\Omega\,t}\right) \tag{22}$$

Applying initial condition $J(0) = J_0$

$$J\,(0) = C_1 + C_2 = J_0 \;\Longrightarrow\; C_2 = -C_1 + J_0 \tag{23}$$

Since only one initial condition is given, one integral constant remains arbitrary.

$$J\,(t) = e^{-R\,t/2\,L}\left(2\,i\,C_1\,\sin\Omega t + J_0\,e^{i\,\Omega\,t}\right) = \mathbf{e^{-R\,t/2\,L}\left(J_0\cos\Omega t + A\sin\Omega t\right)} \tag{24}$$

where $A = 2\,i\,(C_1 + J_0)$ is an arbitrary real constant.

Check it with Mathematica

$$\mathrm{Assuming}\left[\left(\frac{R}{2\,L}\right)^2 - \frac{1}{CL} < 0,\right.$$
$$\left.\mathrm{sol} = \mathrm{DSolve}\left[\left\{L\,j''[t] + R\,j'[t] + \frac{j[t]}{C} == 0,\; j[0] == J_0\right\},\; j[t],\; t\right]\right]$$

$$\left\{\left\{j[t] \to e^{\frac{1}{2}\left(-\frac{R}{L} - \frac{\sqrt{-4\,L + C\,R^2}}{\sqrt{C}\;L}\right)t}\,C[1] - e^{\frac{1}{2}\left(-\frac{R}{L} + \frac{\sqrt{-4\,L + C\,R^2}}{\sqrt{C}\;L}\right)t}\,C[1] + e^{\frac{1}{2}\left(-\frac{R}{L} + \frac{\sqrt{-4\,L + C\,R^2}}{\sqrt{C}\;L}\right)t}\,J_0\right\}\right\}$$

$$\left\{\left\{j[t] \to e^{\frac{1}{2}\left(-\frac{R}{L}-\frac{\sqrt{-4L+CR^2}}{\sqrt{C}L}\right)t} C[1] - e^{\frac{1}{2}\left(-\frac{R}{L}+\frac{\sqrt{-4L+CR^2}}{\sqrt{C}L}\right)t} C[1] + e^{\frac{1}{2}\left(-\frac{R}{L}+\frac{\sqrt{-4L+CR^2}}{\sqrt{C}L}\right)t} J_0\right\}\right\} /. \frac{\sqrt{-4L+CR^2}}{\sqrt{C}L} \to 2 I \Omega$$

$$\left\{\left\{j[t] \to e^{\frac{1}{2}t\left(-\frac{R}{L}-2 i \Omega\right)} C[1] - e^{\frac{1}{2}t\left(-\frac{R}{L}+2 i \Omega\right)} C[1] + e^{\frac{1}{2}t\left(-\frac{R}{L}+2 i \Omega\right)} J_0\right\}\right\}$$

$$\text{sol} = \left\{\left\{j[t] \to e^{\frac{1}{2}t\left(-\frac{R}{L}-2 i \Omega\right)} C[1] - e^{\frac{1}{2}t\left(-\frac{R}{L}+2 i \Omega\right)} C[1] + e^{\frac{1}{2}t\left(-\frac{R}{L}+2 i \Omega\right)} J_0\right\}\right\} \text{ // FullSimplify}$$

$$\left\{\left\{j[t] \to e^{-\frac{t(R+2 i L\Omega)}{2L}} \left(C[1] + e^{2 i t \Omega}\left(-C[1] + J_0\right)\right)\right\}\right\}$$

Take-Home Eaxm

1

(a) The equation of motion is given by

$$m \frac{d^2 x}{dt^2} = -\frac{d}{dx} U(x) = x - x^2$$

Introducing velocity

$$v = \frac{dx}{dt}$$

The equation motion can be written as a pair of first order ODEs.

$$\frac{dx}{dt} = v \equiv f(x, v)$$

$$\frac{dv}{dt} = x - x^2 \equiv g(x, v)$$

Hence, the nullcline for the first equation is

$$f(x, v) = v = 0$$

and for the second equation

$$g(x, v) = x - x^2 = x(x-1) = 0 \Longrightarrow x = 0, \quad x = 1$$

In total there are three nullclines,

$v = 0, \ x = 0,$ and $x = 1$

(b) The fixed points can be obtained from the crossing points of the nullclines

$(x, v) = (0, 0)$ and $(1, 0)$

(c) For expanding f(x,v) and g(x,y) around (0,0), the linearized equations are

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = x$$

Hence, the Jacobian matrix is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

and its eigenvalues are

```
Eigenvalues[A]
```

$\{-1, 1\}$

Since both are real with different signs, the fixed point is **saddle node**.

```
sol = Eigenvectors[A]
```

{{-1, 1}, {1, 1}}

The first eigenvector corresponds to the negative eigenvalue, it represents a stable manifold.

```
u[1] = sol[[1]]
```

{-1, 1}

The second eigenvector represents an unstable manifold.

```
u[2] = sol[[2]]
```

{1, 1}

For the fixed point at (1,0), the first linearized ODE remain the same. Expanding g(x,v) around x=1,

$g(x, v) = g(1, 0) + (x-1)\, g'(1, 0) = -2 (x-1)$

Defining displacement s=x-1, the linearized ODEs are

$\frac{ds}{dt} = v$, and $\frac{dv}{dt} = -2s$

Hence, the Jacobian matrix is

$A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix};$

Its eigenvalues are

```
Eigenvalues[A]
```

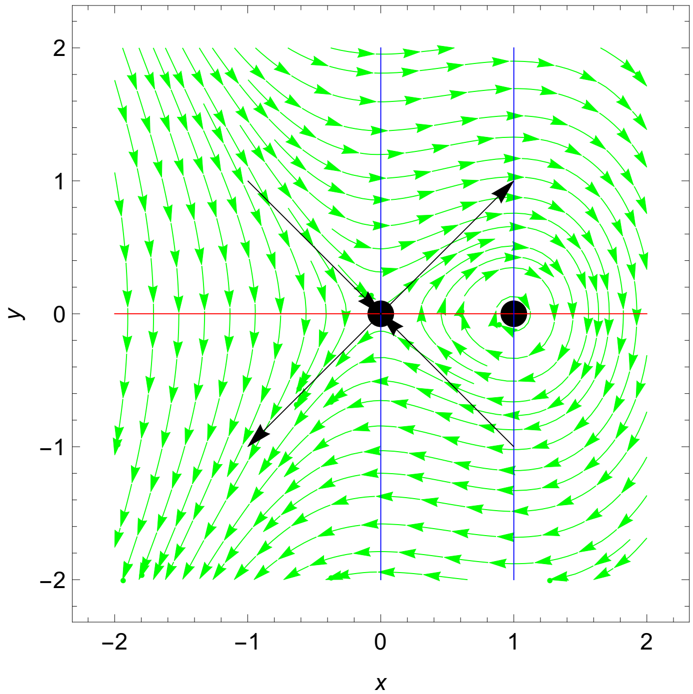
$\{i\sqrt{2}, -i\sqrt{2}\}$

Since both are pure imaginary, the fixed point is **center**

(e)

```
nullcline1 = Graphics[{Red, Line[{{-2, 0}, {2, 0}}]}];

nullcline2 = Graphics[{Blue, {Line[{{{0, -2}, {0, 2}}, {{1, -2}, {1, 2}}}]}}];

fixedpoints = Graphics[{Black, {Disk[{0, 0}, 0.1], Disk[{1, 0}, 0.1]}}];

flows = StreamPlot[{v, x - x^2}, {x, -2, 2}, {v, -2, 2}, StreamStyle → Green, StreamScale →
    {Automatic, Automatic, 0.02}, FrameLabel → {x, y}, BaseStyle → Medium];

arrow1 = Graphics[{Black, Arrow[{{-1, 1}, {0, 0}}], Arrow[{{1, -1}, {0, 0}}]}];

arrow2 = Graphics[{Black, Arrow[{{0, 0}, {1, 1}}], Arrow[{{0, 0}, {-1, -1}}]}];
```

```
Show[{flows, fixedpoints, nullcline1, nullcline2, arrow1, arrow2}]
```



(e)

In the phase portrait, there are three different kinds of trajectories, one is closed orbits around (0,0), the second one which scattered back by the saddle points, and the last one which flows over the saddle points and scattered back by the potential at negative x.
In the following potential diagram, the closed orbit corresponds to the blue line which indicates an oscillatory motion. The orbit scattered by the saddle points are shown in green lines which comes in from left (right) and bounces back. The red line corresponds to phase trajectories that go over the potential maximum and then bounces back.

```
pot = Plot[1/2 x^2 - 1/3 x^3, {x, -1.0, 2.0}, PlotRange → {-0.1, 0.4}];

Solve[1/2 x^2 - 1/3 x^3 == 1/4, x, Reals] // N
```

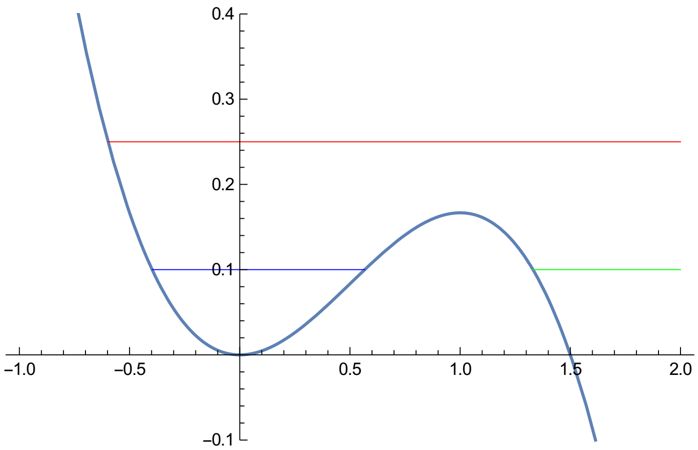
```
{{x → -0.597912}}
```

```
mot1 = Graphics[{Red, Line[{{+2.0, 1/4}, {-0.5979116727228236, 1/4}}]}];

Solve[1/2 x^2 - 1/3 x^3 == 1/10, x, Reals] // N
```

```
{{x → -0.39761}, {x → 0.567069}, {x → 1.33054}}
```

```
mot2 =
  Graphics[{Blue, Line[{{-0.39760987462256187, 1/10}, {0.5670689228522683, 1/10}}]}];

mot3 = Graphics[{Green, Line[{{2.0, 1/10}, {1.3305409517702937, 1/10}}]}];
```

```
Show[{pot, mot1, mot2, mot3}]
```



2. Separating the variable,

$$\frac{1}{v}\, dv = -\gamma\,(t)\, dt \tag{25}$$

Integrating the both side

$$\ln v = -\int \gamma\,(t)\, dt + C \tag{26}$$

and inverting it we find a general solution.

$$v\,(t) = v\,(0)\, e^{-\int_0^t \gamma\,(s)\, ds} = v_0\, e^{-\gamma_0 \int_0^t \cos^2\,(s)\, ds} \tag{27}$$

Carrying out the integral

```
Integrate[Cos[s]^2, {s, 0, t}]
```

$$\frac{1}{2}\left(t + \text{Cos}[t]\, \text{Sin}[t]\right)$$

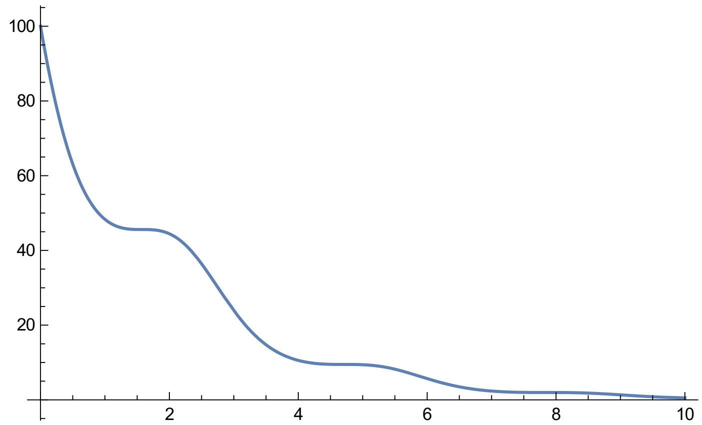
we find the solution

$$v\,(t) = v_0\, e^{-\gamma_0/2\,[t+\sin\,(2\,t)/2]} \tag{28}$$

```
sol = DSolve[{v'[t] == -γ0 Cos[t]^2 v[t], v[0] == v0}, v[t], t] /. {γ0 → 1, v0 → 100}
```

$$\left\{\left\{v[t] \to 100\, e^{-\frac{t}{2}-\frac{1}{4}\text{Sin}[2\,t]}\right\}\right\}$$

```
Plot[v[t] /. sol, {t, 0, 10}]
```



3(a). The Newton equation is written as

$$\frac{dv}{dt} = -g\, e^{-k\,t} \tag{29}$$

and its solution is

$$v\,(t) = v\,(0) - g \int_0^t e^{-k\,s}\, ds = \frac{g}{k}\left(e^{-k\,t} - 1\right) \tag{30}$$

```
sol = DSolve[{v'[t] == -g Exp[-k t], v[0] == 0}, v[t], t] // Simplify
```

$$\left\{\left\{v[t] \to \frac{\left(-1 + e^{-k\,t}\right) g}{k}\right\}\right\}$$

(b)

$$y\,(t) = y\,(0) + \int_0^t v\,(s)\, ds = h_0 + \frac{g}{k}\left(\frac{1}{k}\left(1 - e^{-k\,t}\right) - t\right) \tag{31}$$

```
Integrate[v[t] /. sol, {t, 0, s}]
```

$$\left\{\frac{g\left(1 - e^{-k\,s} - k\,s\right)}{k^2}\right\}$$

(c)

It is clear from Eq (30) that $v \longrightarrow -g/k$.

```
Assuming[k > 0, Limit[v[t] /. sol, t → ∞]]
```

$$\left\{-\frac{g}{k}\right\}$$