Exam 1

1. The ground state is not degenerate. Hence

$$\Delta E_1 = \langle \psi_1 | W | \psi_1 \rangle = \frac{g}{a^2} \int_{-a}^{a} \cos^2(k_1 x)\, \delta(x-b)\, dx \int_{-a}^{a} \cos^2(k_1 y)\, \delta(y)\, dy = \frac{g}{a^2} \cos^2(k_1 b)$$

The remaining states are degenerate. Therefore, we need to diagonalize the sub matrix of W.

$$\langle \psi_2 | W | \psi_2 \rangle = \frac{g}{a^2} \int_{-a}^{a} \sin^2(k_2 x)\, \delta(x-b)\, dx \int_{-a}^{a} \cos^2(k_1 y)\, \delta(y)\, dy = \frac{g}{a^2} \sin^2(k_2 b)$$

$$\langle \psi_2 | W | \psi_3 \rangle = \langle \psi_3 | W | \psi_2 \rangle = \frac{g}{a^2} \int_{-a}^{a} \cos(k_1 x)\, \delta(x-b) \sin(k_2 x)\, dx \int_{-a}^{a} \sin(k_2 y)\, \delta(y) \cos(k_1 y)\, dy = 0$$

$$\langle \psi_3 | W | \psi_3 \rangle = \frac{g}{a^2} \int_{-a}^{a} \cos^2(k_1 x)\, \delta(x-b)\, dx \int_{-a}^{a} \sin^2(k_2 y)\, \delta(y)\, dy = 0$$

The sub matrix is alredy diagonal. Thus $\Delta E_2 = \frac{g}{a^2} \sin^2(k_2 b)$ and $\Delta E_3 = 0$

2.

$$P_{0\to n} = \frac{V_0^2}{\hbar^2} \left| \int_0^t e^{i(E_n - E_0)t/\hbar} \langle n | \hat{a} e^{i\Omega t} + \hat{a}^{\dagger} e^{-i\Omega t} | 0 \rangle\, dt \right|^2$$

$$= \frac{V_0^2}{\hbar^2} \left| \int_0^t e^{i(E_n - E_0 - \hbar\Omega)t} \langle n | \hat{a}^{\dagger} | 0 \rangle\, dt \right|^2 = \frac{V_0^2}{\hbar^2} \left| \int_0^t e^{i(E_n - E_0 - \hbar\omega)t} \delta_{n1}\, dt \right|^2$$

$$\int_0^t e^{i\Delta t}\, dt = \frac{1}{i\Delta} \left[e^{i\Delta t} - 1 \right] \qquad \text{where} \qquad \Delta = \frac{E_n - E - \hbar\omega}{\hbar}$$

$$\left| \frac{1}{i\Delta} (e^{i\Delta t} - 1) \right|^2 = \frac{1}{\Delta^2} (e^{i\Delta t} - 1)(e^{-i\Delta t} - 1) = \frac{1}{\Delta^2} (2 - e^{i\Delta t} - e^{-i\Delta t}) = \frac{2}{\Delta^2} (1 - \cos\Delta t)$$

$$= \frac{4}{\Delta^2} \sin^2 \frac{\Delta t}{2}.$$

$$P_{0\to n} = \frac{4V_0^2}{\hbar^2 \Delta^2} \sin^2 \frac{\Delta t}{2}\, \delta_{n1} = \frac{4V_0^2}{(E_1 - E_0 - \hbar\Omega)^2} \sin^2 \left[(E_1 - E_0 - \hbar\Omega) t / 2\hbar \right] \delta_{n1}$$

$$= \frac{4V_0^2}{\hbar^2 (\omega - \Omega)^2} \sin^2 \left[(\omega - \Omega) t/2 \right] \delta_{n1}$$

$$P_{0\to 1} = \frac{4V_0^2}{\hbar^2 (\omega - \Omega)^2} \sin^2 \left[(\omega - \Omega) t/2 \right] \quad \text{and} \quad P_{0\to 2} = 0$$

3.

Normalization $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = |c|^2 \int_{-\infty}^{\infty} e^{-2ax^2} dx = |c|^2 \sqrt{\frac{\pi}{2a}}$

$$c = \left(\frac{2a}{\pi}\right)^{1/4}$$

Expectation vaule $\langle E \rangle = \int_{-\infty}^{\infty} \psi(x)\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\psi(x)\,dx + \int_{-\infty}^{\infty} |\psi|^2 \frac{m\omega^2}{2} x^2\, dx$

$$\frac{d}{dx}\psi(x) = c\frac{d}{dx} e^{-ax^2} = -2cax\, e^{-ax^2}$$

$$\frac{d^2}{dx^2}\psi(x) = \frac{d}{dx}(-2acx e^{-ax^2}) = -2ac(1-2ax^2)e^{-ax^2}$$

$$\langle E \rangle = \frac{\hbar^2 a}{m}|c|^2 \int_{-\infty}^{\infty} e^{-2ax^2}(1-2ax^2)\,dx + |c|^2 \frac{m\omega^2}{2}\int_{-\infty}^{\infty} x^2 e^{-2ax^2} dx$$

$$= \frac{\hbar^2 a}{2m}|c|^2 \int_{-\infty}^{\infty} e^{-2ax^2} dx + |c|^2\left(\frac{m\omega^2}{2} - \frac{2\hbar^2 a^2}{m}\right)\int_{-\infty}^{\infty} x^2 e^{-2ax^2} dx$$

$$= \frac{\hbar^2 a}{m} + \frac{\cancel{|c|^2}}{4a}\left(\frac{m\omega^2}{2} - \frac{2\hbar^2 a^2}{m}\right)\cancel{\sqrt{\frac{\pi}{2a}}} = \frac{m\omega^2}{8a} + \frac{\hbar^2 a}{2m}$$

$$\frac{\partial\langle E\rangle}{\partial a} = -\frac{m\omega^2}{8a^2} + \frac{\hbar^2}{2m} = 0 \qquad a^2 = \left(\frac{m\omega}{2\hbar}\right)^2 \rightarrow \quad a = \frac{m\omega}{2\hbar}$$

$$E = \frac{m\omega^2}{8\left(\frac{m\omega}{2\hbar}\right)} + \frac{\hbar^2 \frac{m\omega}{2\hbar}}{2m} = \frac{\hbar\omega}{4} + \frac{\hbar\omega}{4} = \frac{\hbar\omega}{2}$$

$$E = \frac{\hbar\omega}{2}, \quad \psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{\pi\hbar}x^2}$$

Comparing this results and exact eigenstate, the trial function is actually exact.