Momentum Deficit due to Dissipation
- From Hydrodynamics Perspective -
(What is the non-equilibrium force on a Brownian particle and how does the environment generate it?)

Ryoichi Kawai
University of Alabama at Birmingham, USA
kawai@uab.edu

Antoine Fruleux
Ken Sekimoto
Nathan Ridling

Phys. Scripta 86 (2012), 058508
arXiv:1301.7035

Stochastic Thermodynamics (NORDITA, 3/4-3/15, 2013)
Introduction: Langevin force and Stochastic Thermodynamics

Under a certain non-equilibrium condition, the environment exerts a force on a Brownian particle which cannot be explained by the Langeivn theory.
Examples:
  Brownian ratchet, Adiabatic piston,
  Inelastic Piston, Granular ratchet

Momentum Deficit due to Dissipation (MDD) is introduced.

Force caused by MDD explains it all.

The environment adjusts itself to MDD and it is no longer equilibrium.

Sasa’s Paradox
What is the force on a Brownian object exerted by an *environment*?

If the environment is an ideal heat bath (energy reservoir at equilibrium)

\[
M \dot{V} = -\gamma V + \sqrt{2\gamma k_B T} \xi - \nabla U(X) \quad \langle \xi(t) \rangle = 0
\]

\[
M \langle \dot{V} \rangle = -\gamma \langle V \rangle - \langle \nabla U(X) \rangle \quad \langle \xi(t) \xi(t') \rangle = \delta(t-t')
\]

What is the force on a Brownian object exerted by the environment under a non-equilibrium condition?

Is the usual heat bath good enough?
If not, is there a universal environment with *desired properties*?
Stochastic Thermodynamics: Driven Non-equilibrium Processes

\[ M \dot{V} = -\nabla U(X) - \gamma V + \sqrt{2\gamma k_B T} \xi - \nabla U_{\text{ext}}(X, \lambda) \]

Environment

\[ \dot{Q} = (-\gamma V + \sqrt{2\gamma k_B T} \xi) \circ dX \]
(work done by the environment)

External Agent

\[ dW = \frac{\partial U}{\partial \lambda} \circ d\lambda \]

\[ M \dot{V} = -\nabla U(X) - \gamma_1 V + \sqrt{2 \gamma_1 k_B T_1} \xi_1 - \gamma_2 V + \sqrt{2 \gamma_2 k_B T_2} \xi_2 \]

\[ \langle \xi_i(t) \rangle = 0 \]

\[ \langle \xi_i(t) \xi_j(t') \rangle = \delta_{ij} \delta(t-t') \]

\[ dQ_1 = (\gamma_1 V + \sqrt{2 \gamma k_B} T_1 \xi_1) \circ dX \]

\[ dQ_2 = (\gamma_2 V + \sqrt{2 \gamma k_B} T_2 \xi_2) \circ dX \]


Stochastic Thermodynamics: Non-equilibrium Steady State (NESS)
Smoluchowski-Feynman Ratchet

Van den Broeck, R.K. and Meurs(2004), PRL 93 090601
Lagevin Eq. can't see asymmetric situation around the Brownian object. (Coarse graining hides it.)

\[ M \dot{V} = -\gamma V + \sqrt{2\gamma k_B T} \xi + F_{\text{EXT}} \] 
\[ \langle V \rangle = 0, \quad \langle V^2 \rangle = \frac{k_B T}{M} \]
Adding a second “heat bath” illuminates the asymmetry. How?

\[ M \dot{V} = -\gamma_1 V + \sqrt{2 \gamma_1 k_B T_1} \xi_1 + F_{\text{EXT}} \]
Replace a “ideal” heat bath with a “realistic reservoir”.

**Shared pistons**

**Ratchet**

**Adiabatic piston**
Replace a “ideal” heat bath with a “realistic reservoir”.

Shared pistons

\[ T_1, p \quad T_2, p \]

Ratchet

\[ T_1 \]

Adiabatic piston
Granular Brownian objects

**Inelastic piston**

\[ T, p \]

\[ e < 1 \]

Costantini et al. EPL 82, 50008 (2008)

**Granular ratchet**

\[ T \]

\[ e, M \]

\[ e < 1 \]

Costantini et al. PRE 75, 061124 (2007)
Cleuren and Van den Broeck, EPL 77, 50003 (2007)
Talbot et al. PRE 82, 011135 (2010)

Internal degrees act as the second environment.
Introduction: Langevin force and Stochastic Thermodynamics

Under a certain non-equilibrium condition, the environment exerts a force on a Brownian particle which cannot be explained by the Langevin theory.

Examples:
Brownian ratchet, Adiabatic piston, Inelastic Piston, Granular ratchet

Momentum Deficit due to Dissipation (MDD) is introduced.

Force caused by MDD explains it all.

The environment adjusts itself to MDD and it is no longer equilibrium.

Sasa's Paradox
Shared Brownian piston

\[ M \dot{V} = -\gamma_1 V + \sqrt{2 \gamma_1 k_B T_1} \xi_1 - \gamma_2 V + \sqrt{2 \gamma_2 k_B T_2} \xi_2 \]

\[ M \langle \dot{V} \rangle = - (\gamma_1 + \gamma_2) \langle V \rangle, \quad \langle V \rangle \to 0 \]

Heat flow through the fluctuation of the pistons' velocity.

\[ \frac{M}{2} \langle V^2 \rangle = \frac{k_B T_{\text{kin}}}{2}, \quad \text{where} \quad T_{\text{kin}} = \frac{\gamma_1 T_1 + \gamma_2 T_2}{\gamma_1 + \gamma_2} \]

\[ J_{\text{diss}} = \frac{k T_1 - k T_2}{M (\gamma_1^{-1} + \gamma_2^{-1})} \quad \text{Smiley Face: Stochastic Thermodynamics} \]

Brownian Motor

It moves!

\[ \langle V \rangle > 0 \quad (T_1 < T_2) \]
\[ \langle V \rangle < 0 \quad (T_1 > T_2) \]

\[ M \dot{V} = -\gamma_1 V + \sqrt{2} \gamma_1 k_B T_1 \xi_1 - \gamma_2 V + \sqrt{2} \gamma_2 k_B T_2 \xi_2 \]
\[ M \langle \dot{V} \rangle = - (\gamma_1 + \gamma_2) \langle V \rangle, \quad \langle V \rangle \to 0 \]

Boltzmann-master eq. predicts this additional force.

\[ \frac{M}{2} \langle V^2 \rangle = \frac{k_B T_{\text{kin}}}{2} \]
\[ c \left( \frac{k_B T_{\text{kin}}}{M} - \frac{k_B T_1}{M} \right) \propto J_{\text{diss}} \]

Van den Broeck, R.K. and Meurs(2004), PRL 93 090601
**Adiabatic Piston**

\[ (T_1, p_1, n_1) \rightarrow (T_2, p_2, n_2) \]

\[ p_1 = p_2, \quad T_1 > T_2, \quad n_1 < n_2 \]

\[ p_i = n_i k_B T_i \]

**Inelastic Piston**

\[ (T, p, n) \rightarrow (T, p, n) \]

\[ e' < e \]

Granular ratchet
Introduction: Langevin force and Stochastic Thermodynamics

Under a certain non-equilibrium condition, the environment exerts a force on a Brownian particle which cannot be explained by the Langeivn theory.

Examples:
- Brownian ratchet, Adiabatic piston,
- Inelastic Piston, Granular ratchet

Momentum Deficit due to Dissipation (MDD) is introduced.

Force caused by MDD explains it all.

The environment adjusts itself to MDD and it is no longer equilibrium.

Sasa's Paradox
Is force on a wall $F=pA$?

$p = \text{hydrostatic pressure}$

\[
\langle V \rangle = 0
\]

Thermally Equilibrium

\[
p = n k_B T
\]
\[
\Delta P = 2 m v_x
\]
\[
\Delta E = 0
\]
\[
F = p A
\]

Thermally Non-equilibrium

\[
p = n k_B T
\]
\[
\Delta P = m v_x + m v'_x
\]
\[
\Delta E \neq 0
\]
\[
F \neq p A
\]
Macroscopic example: Radiometer

\[
\langle \Delta E \rangle = 0
\]
\[
\langle \Delta P \rangle = 2m \langle v_x \rangle
\]

\[
\langle \Delta E \rangle > 0
\]
\[
\langle \Delta P \rangle = m \langle v_x \rangle + m \langle v'_x \rangle > 2m \langle v_x \rangle
\]

J. Clerk Maxwell
Phil. Trans. R. Soc. Lond. 170 (1879), 231
Momentum Deficit due to Dissipation (MDD)

(a) Equilibrium

\[ \sum m v_{in} - \sum (-m v_{out}) \]

Equilibrium Force

\[ F_{ext} \]

(b) Non-equilibrium

\[ \sum m v_{in} - \sum (-m v_{out}) \]

Non-equilibrium Force

\[ F_{ext} \]

\[ \omega_{col} = \text{collision frequency} \]

\[ -F = (m v_{th} + m |v'|) \omega_{col} = (2m v_{th} + m |v'| - m v_{th}) \omega_{col} = pA + F_{MDD} \]

\[ F_{MDD} = (m |v'| - m v_{th}) \omega_{col} \]

\[ pA = 2m v_{th} \omega_{col} \]

\[ \left( \frac{1}{2} m v_{th}^2 - \frac{1}{2} m |v'|^2 \right) \omega_{col} = J_{diss} \]

\[ v_{th} \sim |v| \]

\[ (m v_{th} - m |v'|) \omega_{col} \approx \frac{J_{diss}}{v_{th}} \]

This agrees with the result of lengthy calculation of Boltzmann-Master eq.

\[ F_{MDD} \approx -c \frac{J_{diss}}{v_{th}} \]

\[ c = \sqrt{\frac{\pi}{8}} \text{ for hard disk gas} \]
Claim

- When heat dissipates through the motion of a Brownian object, the environment exerts a force on the Brownian object.
- Its direction is opposite to the direction of the heat flux.
- Its magnitude is proportional to the heat flux, more specifically

\[ F_{\text{MDD}} \approx -c \frac{J_{\text{diss}}}{\nu_{\text{th}}} \quad c > 0 \]

\[ M\langle \dot{V} \rangle = -\left( \gamma_1 + \gamma_2 \right)\langle V \rangle - c \frac{J_{\text{diss}}}{\nu_{\text{th}}} \]
Simple Model 1: Shared Brownian Piston

\[ F_{\text{MDD}} = -\sqrt{\frac{\pi}{8}} \frac{J_{\text{diss}}}{v_{\text{th}}} \]

\[ J_{\text{diss}} = \sqrt{\frac{\pi}{8}} \frac{k T_1 - k T_2}{M \left( \gamma_1^{-1} + \gamma_2^{-1} \right)} \]

\[ \frac{F_{\text{MDD}}}{L} = -\frac{2 \rho_1 \rho_2}{\rho_1 + 2 \rho_2} \frac{m}{M} (k T_1 - k T_2) \]
Simple Model 2: Inelastic Brownian Piston

House keeping dissipation

\[ J_{\text{diss, hk}} = (1 - e) \sqrt{\frac{2}{\pi}} n_{\text{th}} p L \]

Excess dissipation

\[ J_{\text{diss, ex}} = (1 - e) \frac{\gamma}{M} n_{\text{th}} p L \]

\[ \frac{F_{MDD, hk}}{L} = -\frac{1}{2} (1 - e) p \]

\[ \frac{F_{MDD, ex}}{L} = -\frac{m}{M} (1 - e) p \]

\[ e = \text{restitution coefficient} \]
Introduction: Langevin force and Stochastic Thermodynamics

Under a certain non-equilibrium condition, the environment exerts a force on a Brownian particle which cannot be explained by the Langeivn theory.

Examples:
- Brownian ratchet, Adiabatic piston,
- Inelastic Piston, Granular ratchet

Momentum Deficit due to Dissipation (MDD) is introduced.

Force caused by MDD explains it all.

The environment adjusts itself to MDD and it is no longer equilibrium.

Sasa's Paradox
Adiabatic Piston

$p_1 = p_2 \ , \ T_1 > T_2 \ , \ n_1 < n_2$

$$F_{\text{MDD},1} = -\sqrt{\frac{\pi}{8}} \frac{J_{\text{diss}}}{v_{\text{th},1}} \quad F_{\text{MDD},2} = \sqrt{\frac{\pi}{8}} \frac{(-J_{\text{diss}})}{v_{\text{th},2}}$$

$$F_{\text{NET}} = -\sqrt{\frac{\pi}{8}} J_{\text{diss}} \left( \frac{1}{v_{\text{th},2}} + \frac{1}{v_{\text{th},1}} \right) = \sqrt{\frac{\pi}{8}} \frac{k T_1 - k T_2}{M \left( \gamma_1^{-1} + \gamma_2^{-1} \right)} \left( \frac{1}{v_{\text{th},1}} + \frac{1}{v_{\text{th},2}} \right)$$

The piston moves in the opposite direction of the heat.
Granular Brownian Ratchet

\[ F_{\text{BASE}} = P_L - \frac{1-e}{2} P_L - \frac{m}{M} (1-e) P_L \]

\[ F_{\text{SIDE}} \approx -P_L + \frac{1-e}{2} P_L + o(\theta) \quad \text{(Fluctuation is negligible)} \]

\[ F_{\text{NET}} = -\frac{m}{M} (1-e) P_L < 0 \]

Cleuren and Van den Broeck, EPL 77 (2007) 50003
Introduction: Langevin force and Stochastic Thermodynamics

Under a certain non-equilibrium condition, the environment exerts a force on a Brownian particle which cannot be explained by the Langevin theory.

Examples:
- Brownian ratchet
- Adiabatic piston
- Inelastic Piston
- Granular ratchet

Momentum Deficit due to Dissipation (MDD) is introduced.

Force caused by MDD explains it all.

The environment adjusts itself to MDD and it is no longer equilibrium.

Sasa’s Paradox
What kind of reservoir provides $F_{\text{MDD}}$?

\[ M \langle \dot{V} \rangle = - \left( \gamma_1 + \gamma_2 \right) \langle V \rangle - c \frac{J_{\text{diss}}}{v_{\text{th}}} \]

\[ M \dot{V} = - \gamma_1 V + \sqrt{2} \gamma_1 k_B T_1 \xi_1 - \gamma_2 V + \sqrt{2} \gamma_2 k_B T_2 \xi_2 + \text{Non-linear?} \]

Plyukhin and Schofield, PRE 69, 021112 (2004)
Reservoir as a heat bath

\[
\kappa_1 \frac{T_H - \tilde{T}_H}{L_1} = \frac{k_B (\tilde{T}_H - \tilde{T}_C)}{M (\gamma_1^{-1} + \gamma_2^{-1})} = \kappa_2 \frac{\tilde{T}_C - T_C}{L_1}
\]

Common idealization \( \kappa \to \infty \quad (T_H = \tilde{T}_H, T_C = \tilde{T}_C) \)
Momentum flux injected by gravity

Direction of flow

Momentum flux injected by hand

\[ \pi_{ii} = \pi_{ii}^+ + \pi_{ii}^- = 2\pi_{ii}^+ = \frac{-mg}{A} \]

Equilibrium

\[ \pi_{ii}^+ = \pi_{ii}^- \]

Momentum Deficit

\[ \pi_{ii} = \pi_{ii}^+ + \pi_{ii}^- = 2\pi_{ii}^+ - (\pi_{ii}^+ - \pi_{ii}^-) \]

\[ \pi_{ii}^+ \neq \pi_{ii}^- \]
What is happening in the reservoir?

Locally Equilibrium

Non-equilibrium Steady State

Environment must provide both energy and momentum flux that match to the MDD at the piston.

$F = pA$

$p = nk_B T$
Momentum Deficit due to Dissipation in Hydrodynamics

\[
\rho = \int m f(x, c) dc \\
\rho v_i = \int m c_i f(x, c) dc = 0 \\
\rho u = \int \frac{m}{2} c \cdot c f(x, c) dc = \frac{3}{2} n k_B T \\
\pi_{ij} = \int m c_i c_j f(x, c) dc \\
q_i = \int \frac{m}{2} c \cdot c c_i f(x, x) dc
\]

mass density  
momentum density  
energy density  
pressure tensor (momentum flux)  
heat flux

\[
f(x, c) = n \left( \frac{\beta}{\pi} \right)^{3/2} e^{-\beta \cdot c} \left[ 1 + \frac{\beta}{p} \left( c \cdot (\tilde{\pi} + 4 \frac{\beta}{5} q \otimes c) \cdot c - 2 q \cdot c \right) \right]
\]

\[
\beta \equiv \frac{m}{2 k_B T}
\]

\[
\begin{align*}
\pi_{ii}^+ &= \int_{c_i > 0} m c_i c_i f(x, c) dc = \frac{\pi_{ii}}{2} + \frac{1}{5} \sqrt{\frac{2}{\pi}} \frac{q_i}{v_{th}} \\
\pi_{ii}^- &= \int_{c_i < 0} m c_i c_i f(x, c) dc = \frac{\pi_{ii}}{2} - \frac{1}{5} \sqrt{\frac{2}{\pi}} \frac{q_i}{v_{th}} \\
\pi_{ii}^- - \pi_{ii}^+ &= -\frac{1}{5} \sqrt{\frac{8}{\pi}} \frac{q_i}{v_{th}}
\end{align*}
\]

\[
F_{MDD} \approx -c \frac{J_{\text{diss}}}{v_{th}}
\]
Knudsen gas $L < \lambda$

\[
f_j(v) = \begin{cases} 
\rho_j^+ \sqrt{\frac{2m}{\pi k_B T_j}} e^{-m v^2 / k_B T_j}, & \text{v = to piston} \\
\rho_j^- \sqrt{\frac{2m}{\pi k_B T_j}} e^{-m v^2 / k_B T_j}, & \text{v = from piston}
\end{cases}
\]

\[
\rho_j^+ + \rho_j^- = \rho_j
\]

\[
T_j^- \sim T_j + \frac{4 \epsilon}{(1 + \epsilon)^2} (T_{\text{kin}} - T_j)
\]

\[
\epsilon = \frac{m}{M}
\]

Steady state conditions:

\[
J_{\text{mass}} = 0
\]

\[
J_j^E = J_{\text{diss}} = \text{constant}
\]

\[
J_j^P = \pi_j^+ + \pi_j^- = \rho_j^+ k_B T_j + \rho_j^- k_B T_j
\]

\[
F_{\text{MDD}} = \frac{J_j^P - J_{j}^P}{A} = \rho_C k_B T_C - \rho_H k_B T_H - \frac{\sqrt{\pi}}{8} J_{\text{diss}} \left( \frac{1}{v_{\text{th}}^H} + \frac{1}{v_{\text{th}}^C} \right)
\]
Introduction: Langevin force and Stochastic Thermodynamics

Under a certain non-equilibrium condition, the environment exerts a force on a Brownian particle which cannot be explained by the Langevin theory.

Examples:
- Brownian ratchet, Adiabatic piston,
- Inelastic Piston, Granular ratchet

Momentum Deficit due to Dissipation (MDD) is introduced.

Force caused by MDD explains it all.

The environment adjusts itself to MDD and it is no longer equilibrium.

Sasa's Paradox
Sasa's paradox of adiabatic piston

Steady state motion

\[ T_1, p \quad T_2, p \]

\[ F = pA \]

\[ \dot{P} = -\gamma_1 \langle V_1 \rangle - \gamma_2 \langle V_2 \rangle \rightarrow -(\gamma_1 + \gamma_2)\langle V \rangle \]
Conclusions

😊 Concept of Momentum Deficit due to Dissipation (MDD) is introduced.

😊 Force by MDD

\[ F_{\text{MDD}} = -c \frac{J_{\text{diss}}}{V_{\text{th}}} \]

😊 MDD captures the asymmetry in fluctuation.

😊 Adiabatic piston, Brownian ratchets, Inelastic piston, and Granular ratchet ... can be all intuitively and quantitatively explained by MDD without lengthy calculation.

😊 The environment adjusts itself to MDD and becomes non-equilibrium.

😢 We don't know a simple stochastic model for such an environment. (Does it exist?)